Studies on Blow-up

by

Anwar A. Hussain Saleh

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
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This thesis, written by

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under the direction of his thesis advisor and approved by the thesis committee has been presented to and accepted by the Dean College of Graduate Studies, in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE IN MATHEMATICS

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Date: ١٤١٤ - ٩ - ١٣

خلاصة الرسالة

اسم الطالب : أنور عبدالحسين صالح محمد
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ان معادلات التفاعل غير الخطية في الغالب ان وجد لها حلا اعتياديا ، مع نوعية مناسبة من المعلومات الاولية ، فهذا الحل يكون لفترة محدودة من الوقت مثلا [٠ ، ا) ، وعندما يقترب الوقت من النقطة أ ، فان الحل الاعتيادي يقترب من اللانهاية و عندها تسمى النقطة أ نقطة لاتناهي للحل الاعتيادي .

فهذه الرسالة تبحث في نوعية الدوال غير الخطية الداخلة في تلك المعادلات ، مع نوعية مناسبة من المعلومات الاولية في المجال اللامتناهي بحيث أن الحل الاعتيادي يتواجد في كل الاوقات في الفترة [٠ ،∞) ويصبح لامتناهيا عند نقطة اللاتناهي للمجال .

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Thesis Abstract

It is known that the solution of the nonlinear reaction–diffusion equation

$$\begin{aligned} u_t &= \Delta u + u^\lambda \quad x \in R^n, \ 1 < \lambda < \frac{n+2}{n}, \quad n > 3 \\ u(0,x) &= u_9(x) \quad x \in R^n. \end{aligned}$$

becomes unbounded in finite time no matter how small the initial data u_0 is. This time is known as the finite blow–up time.

In this thesis, we push the blow–up time into infinity for the general equation by introducing a cooling term, that is the equation

$$\begin{aligned} u_t &= \Delta u + f(u,x) - \frac{2|\nabla u|^2}{(u+1)} \quad x \in R^n, \ t > 0 \\ u(0,x) &= u_0(x) \quad x \in R^n \end{aligned}$$

Also, the infinite time blow–up of $u(x,t)$ will occur at the infinity; $|x| = \infty$. R^n.

Summary

Chapter 0 Contains historical remarks and some illustrative examples and concepts of blow–up notion.

Chapter I is a summary of semigroup approach for semilinear parabolic equations and some sufficient conditions for "Global existence".

Chapter II Deals with a class of semilinear parabolic equations "u–equation". Under certain substitution this class is changed into another class of semilinear parabolic equation "v–equation" for which existence–uniqueness are shown. The convergence of t–dependent solution into a decaying steady–state solution is given. Also, asymptotics for the mentioned convergence and some examples of the main result are given for both equations (U and V).

Chapter III Deals with the same problems in chapter II but for higher dimensions; $n \geq 3$, are the main result.

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CHAPTER 0 Introduction and Illustrative Examples

§1. Introduction.

The evolution equations (reaction–diffusion, Schrödinger, wave) can be considered to encompass a general class of important equations of mathematical physics with the variable t, which are needed to describe a process evolving with time in physics, mechanics and some other applied mathematics fields for linear equation with smooth enough initial data, we expect the Cauchy problem to admit a unique global solution with suitable regularity on $t \geq 0$. The situation is not the same for nonlinear evolution equations. Generally speaking, for nonlinear ones, classical solutions exist only locally in time even if the initial data are sufficiently smooth and small. Here are some examples.

§2. Examples

Example (0.2.1)

Consider the nonlinear ordinary differential equation

$$\frac{du}{dt} = u^2$$

where the solution is

$$u = \frac{u_0}{1 - u_0 t}$$

So if $u_0 > 0$

$$u \to \infty \text{ as } t \to \frac{1}{u_0}$$

i.e., the solution $u(t)$ blows up at $t = 1/u_0$. Thus for (0..2.1), we can only obtain the local solution on $(0, \frac{1}{u_0})$. Formal definition of blow–up will be given later.

Example: (0.2.2).

Consider the semilinear parabolic equation

$$u_t = \Delta u + u^2 \qquad \text{on} \qquad D = (0, \infty) \times \Omega \tag{0.2.3}$$

$$\begin{aligned} \frac{\partial}{\partial n} u(t,x) &= 0 \text{ on } \partial\Omega \\ u(0,x) &= u_0 \quad x \in \Omega \end{aligned} \tag{0.2.4}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$.

Let $\int_\Omega u_0(x)dx > 0$. Then (0.2.4) does not admit a classical global solution on D. To see this, let

$$U(t) = \int_\Omega u(t,x)dx$$

integrating (0.2.4) with respect to x and using Green's theorem, with the described boundary condition yields

$$\frac{dU(t)}{dt} = \int_\Omega u^2(t,x)dx$$

By Holder inequality

$$U(t) \leq \left(\int_\Omega u^2(t,x)dx \right)^{1/2} |\Omega|^{1/2}$$

which gives

$$\begin{aligned} \frac{dU}{dt} &\geq \frac{U^2(t)}{|\Omega|} \\ U(0) &= \int_\Omega u_0(x)dx > 0 \end{aligned}$$

Thus by the above example, since (if $V(t)$ is the solution as in (0.2.1)) $V(t) \leq U(t)$ which shows that $U(t)$ blows up, i.e. the global solution does not exist.

Example: (0.2.5)

$$\begin{aligned} u_t &= \Delta u + u^{1+\alpha} \qquad x \in R^n, \quad t > 0 \\ u(x,0) &= u_0(x) \end{aligned}$$

The classical solution was shown by (Fujita) [6] to blow up (i.e. become unbounded; formal definition will be given later) if

$$\alpha \leq \frac{2}{n} \qquad n \geq 3.$$

These examples show the importance of existence of classical global solutions. There are some methods to obtain the global classical solution, one of them is the semigroups of linear operator which reduces the PDE into an ODE in Banach spaces. Another method is upper and lower solutions. Both methods are discussed.

CHAPTER I

General Theory of Semigroups Approach

§1. Introduction:

Consider the semilinear parabolic equation

$$\begin{aligned} Lu \equiv u_t - \Delta u &= f(x,u) \quad t > 0, x \in R^n \\ u(x,0) &= \phi(x) \quad x \in R \end{aligned} \tag{1.1.1}$$

Definition (1.1.2) (Upper Solution)

A function $w \in C^{2,1}$ $(D_T = \Omega \times (0,T)$ in an upper solution of (1.1.1) if it satisfies

$$\begin{aligned} Lw &\le f(x,w) \quad \text{in } D_T \\ w(x,0) &\ge \phi(x) \quad \text{on } \partial\Omega \end{aligned} \tag{1.1.3}$$

with the growth condition

$$|w(x,t)| \le C_0 \exp\{b|x|^2\} \quad \text{as } |x| \to \infty. \tag{1.1.4}$$

Definition (1.1.5) (Lower Solution)

A function $v \in C^{1,2}(D_T)$ is a lower solution if it satisfies

$$\begin{aligned} Lv &\ge f(x,v) \quad (t,x) \in D_T \\ v(x,0) &\le \phi(x) \end{aligned} \tag{1.1.6}$$

with the growth condition for some w equal or smaller than C_0.

Theorem 1.1.7. Let v, w be lower and upper solutions (resp.) of (1.1.1), and $f(v, x)$ satisfies

(i) $f(0, x) \geq v_0 \geq 0$.

(ii) for $v \leq v_2 \leq v_1 \leq w$, there exists C_1, C_2 continuous bounded functions such that

$$C_1(v_1 - v_2) \leq f(v_1, x) - f(v_2, x) \leq C_2(v_1 - v_2)$$

(ii) For some $z \in (v, w)$

$$|f(z, x)| \leq A_0 \exp\{b|x|^2\} \qquad \text{as } |x| \to \infty$$

where A_0 and b are positive constants. Then equation (1.1.1) has a unique solution v such that $v \leq u \leq w$.

Definition 1.1.8 The solution $u(x, t)$ of (1.1.1) is said to blow–up at (x_0, T) if there is a sequence (x_n, T_n) so that $u(x_n, T_n) \to \infty$ as $(x_n, T_n) \to (x_0, T)$.

§2 Semigroups

Definition (1.2.1) We call a linear operator A in a Banach space X a sectorial operator if it is a closed densely defined operator such that for some ϕ in $(0, \pi/2)$ and some $M \geq 1$ and real a, the sector

$$S_{a,\phi} = \{\lambda : \phi \leq |\arg(\lambda - a)| \leq \pi, \quad \lambda \neq a\}$$

is in the resolvent set of A and

$$|(\lambda - A)|^{-1} \leq M|\lambda - a| \text{ for all } \lambda \in S_{a,\phi}$$

Examples (1.2.2)

1. If A is bounded linear operator on a Banach space, then A is sectorial.
2. If A is self-adjoint operator densely defined on a Hilbert space and if A is bounded from below, then A is sectorial.

Definition (1.2.3) An analytic semigroup on a Banach space X is a family $\{T(t)\}_{t\geq 0}$ of continuous linear operators on X, satisfying

(i) $T(0) = I, \;\; T(t)T(s) = T(t+s)$ for $t, s \geq 0$

(ii) $T(t)x \to x$ as $t \to 0^+$ for each $x \in X$

(iii) $t \to T(t)x$ is real analytic on $0 < t < \infty$ for each $x \in X$.

Definition (1.2.4) For $h > 0$, introduce

$$A_h x = \frac{T(h)x - x}{h} \quad (x \in X)$$

and denote by D_A the set of all $x \in X$ for which $\lim_{h \to 0} A_h x$ exists. Define the operator A with domain D_A by

$$Ax = \lim_{h \to 0} A_h x$$

We call A the infinitesimal generator of the semigroup $\{T(t)\}$. We usually write $T(t) = e^{At}$ if A is continuous with respect to t.

Theorem (1.2.5) If A is a sectorial operator, then $-A$ is the infinitesimal generator of analytic semigroup $\{e^{-tA}\}_{t \geq 0}$, where

$$e^{-At} = \frac{1}{2\pi i} \int_\tau (\lambda + A)^{-1} e^{\lambda t} \, d\lambda$$

where τ is a contour in $\rho(-A)$ with $arg\lambda \to \pm\theta$ as $|\lambda| \to \infty$ for some θ in $(\pi/2, \pi)$.

Further e^{-At} can be continued analytically into a sector $\{t \neq 0 : |\arg t| < \epsilon$ containing the positive real axis, and if $\sigma(A) > a$, then for $t > 0$

$$\|e^{-At}\| \leq Ce^{-at}$$

$$\|Ae^{-At}\| \leq \frac{C}{t} e^{-at}$$

for some constant C.

Finally,

$$\frac{d}{dt} e^{-At} = -Ae^{-At} \quad \text{for } t > 0$$

Proof: Without loss of generality, assume $a > 0$ and $M(\lambda + A)^{-1}\| \leq M/|\lambda| + \delta$ for $|\pi - \arg\lambda| \geq \phi$ for some constant $\delta > 0$, $M > 0$ and $\phi \in (0, \pi/2)$; otherwise replace A by $A - aI$.

Choose θ in $\pi/2 < \theta < \pi - \phi$. Define e^{-At} by the above integral, and note that the integral converges absolutely if $t > 0$. By Cauchy theorem, the integral is unchanged when the contour τ is shifted to the right by a small distance: Call the shifted contour τ' then for $t > 0$, $s > 0$

$$\begin{aligned} e^{-At}e^{-As} &= (2\pi i)^{-2}\int_\Gamma\int_{\Gamma'} e^{\lambda t}(\lambda I + A)^{-1}e^{\mu s}(\mu I + A)^{-1}d\mu d\lambda \\ &= (2\pi i)^{-2}\int_\Gamma\int e^{\lambda t + \mu s}(\mu - \lambda)^{-1}\left\{(\lambda f + A)^{-1} - (\mu I + A)^{-1}\right\}d\mu d\lambda, \quad (1.2.6) \end{aligned}$$

Using the resolvent identity. But for $\lambda \in \Gamma$, $\mu \in \Gamma'$

$$\int_\Gamma e^{\lambda t}(\mu - \lambda)^{-1}d\lambda = 0, \quad \int_{\Gamma'} e^{\mu s}(\mu - \lambda)^{-1}d\mu = 2\pi i e^{\lambda s}$$

So,

$$e^{-At}e^{A}s = (2\pi i)^{-1}\int_\Gamma e^{\lambda(t+s)}(\lambda I + A)^{-1}d\lambda = e^{-A(t+s)}$$

and thus, if $x \in D(A)$, as $t \to 0^+$

$$\frac{1}{t}(e^{-At}x - x) = -\frac{1}{t}\int_0^t e^{-As}Axds \to -Ax$$

and $\{e^{-At}\}_{t\geq 0}$ is a semigroup. In fact, for $0 < \epsilon < \theta - \pi/2$, the integral converges in any compact set of $\{|\arg t| < \epsilon\}$, so the semigroup is analytic there.

Also, putting $\mu = \lambda t$ in the integral { with $t > 0$}

$$\|e^{-At}\| = \|\frac{1}{2\pi i}\int_\Gamma mu(\frac{\mu}{t} + A)^{-1}\frac{d\mu}{t}\|$$

$$\leq \frac{M}{2\pi} \int_T |e^{\mu}| \frac{|d\mu|}{|\mu|}$$

$$\|e^{-At}\| \leq \frac{1}{2\pi} \frac{M}{\delta} \int_T |e^{\mu}| \frac{|d\mu|}{|\mu|} \cdot \frac{1}{t} = \frac{\text{constant}}{t}$$

We now prove $e^{-At}x \to x$ as $t \to 0^+$ for each $x \in X$. It suffices to prove this for $x \in D(A)$, a dense set, since $\|e^{-At}\| \leq C$ for all $t \geq 0$. If $x \in D(A)$ and $t > 0$,

$$\begin{aligned} e^{-At}x - x &= \frac{1}{2\pi i} \int_T e^{\lambda t}[(\lambda I + A)^{-1} - \lambda^{-1}]x d\lambda \\ &= -\frac{1}{2\pi i} \int_T \lambda^{-1} e^{\lambda t} A(\lambda I + A)^{-1} x d\lambda \end{aligned}$$

So

$$\|e^{-At}x - x\| \leq \text{ constant } \|Ax\|t.$$

Thus, $\{e^{-tA}\}_{t \geq 0}$ is a strongly continuous semigroup which extends to an analytic semigroup in $|\arg t| < \epsilon$. If $x \in D(A)$, $t > 0$, then

$$\frac{d}{dt} e^{-At}x + Ae^{-At}x = \frac{1}{2\pi i} \int_T e^{\lambda t}(\lambda + A)(\lambda + A)^{-1} x d\lambda = 0$$

Thus, if $x \in D(A)$, as $t \to 0^+$

$$\frac{1}{t}\left(e^{-At}x - x\right) = -\frac{1}{t} \int_0^t e^{-As} Ax \to -Ax$$

So $-A$ is contained in the generator G of the semigroup.

To see that $-A$ actually is the generator, for $\lambda \geq 0$, define

$$R(\lambda)x = \int_0^{\infty} e^{-\lambda t} e^{-At} x \, dt.$$

For any $x, e^{-At}x \in D(A)$, for $t > 0$ and for $\delta > 0$

$$A \int_\delta^{\infty} e^{-\lambda t} e^{-At} x \, dt = e^{-\lambda\delta} e^{-\delta A} x - \lambda \int_\delta^{\infty} e^{-\lambda t} e^{-At} x \, dt.$$

By the closedness of A, it follows that $R(\lambda)x \in D(A) \subseteq D(G)$ for every $t \geq 0$, $x \in X$, but if $x \in D(G)$, then $e^{-At}x \in D(G)$ for all $t > 0$ and

$$Ge^{-At} = \frac{d}{dt}e^{-At}x = e^{-At}Gx$$

and a similar argument shows

$$R(\lambda)(\lambda - G)x = x \qquad \text{for } x \in D(G)$$

Thus $D(G) \subset$ range of $R(\lambda) \subset D(A)$, hence $-A = G$ as claimed.

§3. Fractional Powers of Operators

Helps solves the integral equation by interpolatin.

Definition (1.3.1) Suppose A is a sectorial operator and Re $\sigma(A) > 0$; then for any $\alpha > 0$

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} dt$$

Example (1.3.2) If A is a positive definite, self-adjoint operator in a separable Hilbert space with spectral representation

$$A = \int_0^\infty \lambda \, dE(\lambda), \qquad \text{then}$$

$$A^{-\alpha} = \int_0^\infty \lambda^{-\alpha} dE(\lambda).$$

Theorem (1.3.3) If A is a sectorial operator in X with Re $\sigma(A) > 0$, then for any $\alpha > 0$, $A^{-\alpha}$ is bounded linear operator on X which is one-to-one and satisfies

$A^{-\alpha}\ A^{-\beta} = A^{-(\alpha+\beta)}$ whenever $\alpha > 0$, $\beta > 0$. Also, $0 < \alpha < 1$.

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda + A)^{-1} d\lambda$$

<u>Proof</u>: For some $\delta > 0$, Re $\sigma(A) > \delta$, so by Th. (1.2.5), $\|e^{-At}\| \leq Ce^{-At}$ for $t > 0$. Thus, $\|A^{-\alpha}x\| \leq \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} Ce^{-\delta t} dt \|x\|$, and $A^{-\alpha}$ is bounded when $\alpha > 0$. Also, for $\alpha > 0$, $\beta > 0$

$$\begin{aligned} A^{-\alpha}A^{-\beta} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-A(t+s)}dsdt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty du \int_0^t t^{\alpha-1}(u-t)^{\beta-1}e^{-Au}dt \\ &= A^{-(\alpha+\beta)}, \text{ using } \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}dz = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \end{aligned}$$

Also, if $A^{-\alpha}x = 0$, for some $\alpha > 0$, then for integer $n > 1$, $A^{-n}x = A^{-(n-\alpha)}A^{-\alpha}x = 0$; but A^{-1} is one-to-one so $A^{-n} = n$-th power of A^{-1} is also one-to-one, so $x = 0$.

Finally,

$$(\lambda + A)^{-1} = \int_0^\infty e^{-At}e^{-\lambda t}dt \qquad \text{for } \lambda \geq 0,$$

so

$$\begin{aligned} \int_0^\infty \lambda^{-\alpha}(\lambda + A)^{-1}d\lambda &= \int_0^\infty e^{-At}\Big(\int_0^\infty \lambda^{-\alpha}e^{-\lambda t}d\lambda\Big)dt \\ &= \int_0^\infty e^{-At}t^{\alpha-1}\Gamma(1-\alpha)dt = \frac{\pi}{\sin \pi\alpha}A^{-\alpha} \end{aligned}$$

using the fact that $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}$ for $0 < \alpha < 1$.

Definition (1.3.4). With A as above, define $A^\alpha =$ inverse of $A^{-\alpha}$ $\quad \alpha > 0$, $D(A^\alpha) = R(A^{-\alpha})$; $A^0 =$ identity on X.

Theorem (1.3.5) Suppose A is sectorial and Re $\sigma(A) > \delta > 0$. For $\alpha \geq 0$ there exists $c_\alpha < \infty$ such that

$$\|A^\alpha e^{-At}\| \leq c_\alpha t^{-\alpha} e^{-\delta t} \quad \text{for } t > 0$$

and if $0 < \alpha \leq 1 \quad t, x \in D(A^\alpha)$

$$\|(e^{At} - I)x\| \leq \frac{1}{\alpha} C_{1-\alpha} t^\alpha \|A^\alpha x\|.$$

Also, C_α is bounded for α in any compact interval of $(0, \infty)$. We see in the proof that C_α is bounded as $\alpha \to 0^+$.

<u>Proof</u> $\|e^{-At}\| \leq Ce^{-At}, \|Ae^{-At}\| \leq Ct^{-1}e^{-At}$ for $t > 0$, by Th. (1.2.5), so for $m = 1, 2, 3, \ldots$

$$\|A^m e^{-At}\| = \|(Ae^{-At/m})^m\| \leq (C_m)^m t^{-m} e^{-\delta t}$$

if $0 < \alpha < 1, \quad t > 0,$

$$\begin{aligned} \|A^\alpha e^{-At}\| &= \|Ae^{-At} \cdot A^{-(1-\alpha)}\| \\ &\leq \frac{1}{\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha} \|Ae^{-A(t+s)}\| ds \\ &\leq Ct^{-\alpha} e^{-\delta t} \Gamma(\alpha) \end{aligned}$$

Finally,

$$\begin{aligned} \|e^{\alpha+\beta} e^{-At}\| &\leq \|A^\alpha e^{At/2}\| \|A^\beta e^{-At/2}\| \\ &\leq C_\alpha C_\beta 2^{\alpha+\beta} t^{-(\alpha+\beta)} e^{-\delta t}, \end{aligned}$$

and putting all these cases together gives the general result. The other estimate follows from

$$(e^{-At} - I)x = -\int_0^t A^{1-\alpha} e^{-As} A^\alpha x ds.$$

Theorem (1.3.6) If $0 \le \alpha \le 1$, $x \in D(A)$, then $\|A^\alpha x\| \le C\|Ax\|^\alpha \|x\|^{1-\alpha}$ i.e., $\|A^\alpha x\| \le \epsilon\|Ax\| + C'\epsilon^{-\alpha/(1-\alpha)}\|x\|$ for all $\epsilon > 0$ (C, C' are constants independent of α)

Proof Let $0 < \beta < 1$, $\epsilon > 0$, so (if $\|e^{-At}\| \le C$ for $t > 0$)

$$\begin{aligned}\|\Gamma(\beta)A^{-\beta}x\| &= \|(\int_0^\epsilon + \int_\epsilon^\infty)t^{\beta-1}e^{-At}x\,dt\| \\ &\le C\|x\|\frac{\epsilon^\beta}{\beta} + \|\epsilon^{\beta-1}e^{-A\epsilon}A^{-1}x + \\ &\qquad (\beta-1)\int_\epsilon^\infty t^{\beta-1}e^{-At}A^{-1}x\| \\ &\le C\|x\|\frac{\epsilon^\beta}{\beta} + 2C\|A^{-1}x\|\epsilon^{\beta-1}\end{aligned}$$

Minimizing the right hand side over $\epsilon > 0$ and

$$\|A^{-\beta}\| \le \frac{2(2(1-\beta))^{\beta-1}}{\Gamma(1+\beta)} C\|x\|^{1-\beta}\|A^{-1}x\|^\beta$$

The coefficient is bounded uniformly on $0 < \beta < 1$, so we can replace x by Ax and set $\alpha = 1 - \beta$ to get the claimed result.

Remark (1.3.7). if

$$\|e^{-At}\| \le CC_0e^{-\delta t}, \quad \|Ae^{-At}\| \le C_1t^{-1}e^{-\delta t}$$

for $t > 0$, then (if $C$is the constant from Th (1.3.6))

$$\|A^\alpha e^{-At}\| \le C_0^{1-\alpha}C_1^\alpha t^{-\alpha}e^{-\delta t},$$

proving C_α in Th (1.3.5) is bounded as $\alpha \to 0^+$.

Definition (1.3.8). Let I be an interval. A function $f : I \to X$ is Holder continuous with exponent $\alpha, 0 < \alpha < 1$ on I if there is a constant L such that

$$\|f(t) - f(s)\| \leq L\|t - s\|^\alpha \quad \text{for } s, t \in I$$

It is locally Holder continuous if every $t \in I$ has a neighborhood in which f is Holder continuous.

Definition (1.3.9). If A is a sectorial operator in a Banach space X, define for each $\alpha \geq 0$ $X^\alpha = D(A_1^\alpha)$ with the graph norm

$$\|x\|_\alpha = \|A_1^\alpha x\| \quad x \in X^\alpha$$

where $A_1 = A + aI$ with a chosen so Re $\sigma(A_1) > 0$. Different choices of a give equivalent norm on X^α (see D. Henry)]8]

Theorem (1.3.10). $(X^\alpha, \|\cdot\|_\alpha)$ as defined above is a Banach space. For $\alpha \geq \beta \geq 0$ X^α is a dense subspace of X^β.

Lemma (1.3.11). Let $f : (0, T) \to X$ be locally Holder continuous with $\int_0^\rho \|f(s)\|ds < \infty$ for some $\rho > 0$. For $0 \leq t < T$, define

$$F(t) = \int_0^t e^{-A(t-s)} f(s)ds.$$

Then $F(\cdot)$ is continuous on $(0, T)$, continuously differentiable on $(0, T)$, with $F(t) \in D(A)$ for $0 < t < T$ and $dF(t)/dt + AF(t) = f(t)$ on $0 < t < T$, $F(t) \to 0$ in X as $t \to 0^+$.

Proof: For small $\rho > 0$ define

$$F_\rho(t) = \int_0^{t-\rho} e^{-A(t-s)} f(s)ds, \quad \rho \le t < T$$

with $F_\rho(t) = 0$ for $0 \le t \le \rho$.

Then (setting $f(s) = 0$ for $s < 0$)

$$\|F(t) - F_\rho(t)\| \le \int_{t-\rho}^{t} \|e^{-A(t-s)}\| \, \|f(s)\| ds$$

which tends to 0 as $\rho \to 0^+$, uniformly in $0 \le t \le t_0$ for any $t_0 < T$. Also, F_ρ is continuous, since,

$$F_\rho(t+h) - F_\rho(t) = (e^{-Ah} - I) \int_0^{t-\rho} e^{-A(t-s)} f(s)ds + \int_{t-\rho}^{t+h-\rho} e^{-A(t+h-s)} f(s)ds$$

($0 \le t \le t+h \le t_0$), which tends to 0 as $h \to 0$. Therefore, F is continuous on $(0,T)$ into X, and

$$\|F(t)\| \le \int_0^t \|e^{-A(t-s)}\| \, \|f(s)\| ds \to 0 \text{ as } t \to 0^+$$

Also, if $0 \le s < t$, then $e^{-A(t-s)} f(s)$ is in $D(A)$, so the Riemann sum for F_ρ

$$\sum_{t-s_j \ge \rho} e^{-A(t-s_j)} f(s_j) \Delta s_j, \text{ are in } D(A), \text{ and}$$

$$\lim_{\Delta s \to 0} A \sum_{s \le t-\rho} e^{-A(t-s)} f(s)ds = \int_0^{t-\rho} Ae^{-A(t-s)} f(s)ds.$$

Thus, by closedness of A, $F_\rho(t) \in D(A)$ and

$$AF_\rho(t) = \int_0^{t-\rho} Ae^{-A(t-s)} f(s)ds = \int_0^{t-\rho} Ae^{-A(t-s)} \{f(s) - f(t)\} ds + \{e^{-A\rho} - e^{-At}\} f(t)$$

Now $\|Ae^{-A(t-s)}\| = O((t-s)^{-1})$, $\|f(s) - f(t)\| = O(|t-s|^\theta)$ for some $\theta > 0$ as $s \to t^-$, hence as $\rho \to 0^+$

$$AF_\rho(t) = \int_0^t Ae^{-A(t-s)} \{f(s) - f(t)\} \, ds.$$

Thus, again by the closedness of $A, F(t) \in D(A)$ for $0 < t < T$; consider any strictly interior interval $[t_0, t_1]$, $0 < t_0 < t_1 < T$; then $AF_\rho(t) \to AF(t)$ uniformly on $t_0 \le t \le t_1$, since

$$\|f(t) - f(s)\| \le K|t-s|^\theta \text{ for } t, s \text{ in } (t_0, t_1]$$

and some $\theta > 0$, so

$$\begin{aligned} \|AF_\rho(t) - AF(t)\| &= \|\{-I + e^{-A\rho}\}f(t) + \int_{t-\rho}^t Ae^{-A(t-s)} \times \\ &\qquad \{f(s) - f(t)\}ds\| \\ &\le \|\{e^{-At} - I\}f(t)\| + C\int_{t-\rho}^t (t-s)^{-1+\theta} ds \to 0 \text{ as } \rho \to 0^+ \end{aligned}$$

uniformly in $t_0 \le t \le t_1$.

Finally, $F_\rho(t)$ is differentiable when $t > \rho$, with

$$\frac{dF_\rho(t)}{dt} = -AF_\rho(t) + e^{-A\rho} f(t-\rho), \quad \rho < t < T.$$

The right hand side converges uniformly to

$$-AF_\alpha(t) + f(t) \text{ on } t_0 \le t \le t_1 \text{ as } \rho \to 0^+,$$

so F is continuously differentiable on the open interval $(0, T)$, with

$$\frac{dF}{dt} + AF = f(t).$$

Theorem (1.3.12) Suppose A is a sectorial operator in X, $u_0 \in X, f : [0,T) \to X$ is locally Holder continuous and $\int_0^\rho \|f(t)\|dt < \infty$ for $\rho > 0$; then there exists a unique solution $u(\cdot)$ of

$$\frac{du}{dt} + Au = f(t) \quad 0 < t < T \quad u(0) = u_0$$

namely

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}f(s)ds$$

Consider the nonlinear equation

$$\frac{du}{dt} + Au = f(t,u) \quad t > t_0 \tag{1.3.13}$$

$$u(t_0) = u_0$$

where A is a sectorial operator so that the fractional powers of $A_1 = A + aI$ are well defined with the graph norm $\|u\|_\alpha = \|A_1^\alpha\|$ are defined for some $\alpha \geq 0$. . Assume f maps some open set U in $R \times X^\alpha$ into X for some $0 < \alpha < 1$, and f is locally Holder continuous in t and locally Lipschitz in x on U.

Definition (1.3.14) A solution of the above problem on (t_0, t_1) is a continuous function $u : (t_0, t_1) \to X$ such that $u(t_0) = u_0$ and on (t_0, t_1) we have $(t, u(t)) \in U$, $u(t) \in D(A)$, $\frac{du}{dt}(t)$ exists, $t \to f(t, u(t))$ is Holder continuous, and $\int_{t_0}^{t_0+\rho} \|f(t, u(t))\|dt < \infty$ for some $\rho > 0$ and the differential equation (1.3.13) is satisfied on (t_0, t_1).

Theorem (1.3.15) If u is a solution of (1.3.13) on (t_0, t_1), then

$$u(t) = e^{A(t-t_0)}u_0 + \int_{t_0}^{t_1} e^{-A(t-s)} f(s, u(s))ds. \tag{1.3.16}$$

conversely, if u is a continuous function from (t_0, t_1) into X^α, and $\int_{t_0}^{t_0+\rho} \|f(s, u(s))ds < \infty$ for some ρ and if the integral equation (1.3.16) holds for $t_0 < t < t_1$, then $u(\cdot)$ is a solution of the differential (1.3.13) on (t_0, t_1).

Proof. The first claim is immediate from the definition of the solution and Th (1.3.12). Suppose u is a solution of (1.3.16) and $u \in C((t_0, t_1);\ X^\alpha)$. We first prove that u is locally Holder continuous from (t_0, t_1) to X^α if $0 < t < t + h < t_1$, then

$$u(t+h) - u(t) = (e^{-Ah} - I)e^{-A(t-t_0)}u_0 + \int_{t_0}^{t}(e^{-At} - I)e^{-A(t-s)}$$
$$f(s, u(s))ds + \int_{t}^{t+h} e^{-A(t+h-s)} f(s, u(s))ds$$

Now, if $0 < \delta < 1 - \alpha$, then for any $z \in X$,

$$\|(e^{-Ah} - I)e^{-A(t-s)}z\|_\alpha \leq C(t-s)^{-(\alpha+\delta)}h^\delta e^{a(t-s)}\|z\|.$$

(by Th (1.3.5)). Hence for $z \in (t_0^*, t_1^*) \subset (t_0, t_1)$

$$\|u(t+h) - u(t)\|_\alpha \leq \text{const.}h^\delta$$

It follows that $t \to f(t, u(t))$ is locally Holder continuous on (t_0, t_1) so by Th. (1.3.12) u solves the linear equation

$$\frac{du}{dt} + Au = f(t, u(t)), \quad u(0) = u_0$$

on $t_0 < t < t_1$. Hence, x is a solution of (1.3.13) on (t_0, t_1)

Remark. Semigroups are applied in case of $\|u\| < \infty$ which is not the case if blow–up occurs.

CHAPTER II

The One–Dimensional Reaction–Diffusion Equation

§1. Statement of the Problems

Consider the semilinear parabolic equation

$$u_t = \Delta u + |u|^{p-1}u - |\nabla u|^q \quad (x,t) \in \Omega \times (0,T) \quad p > 1, q > 1$$

$$u(x,0) = u_0(x) \quad x \in \Omega$$

$$u(x,t) = 0 \quad (x,t) \in \partial\Omega \times (0,T).$$

where Ω is a bounded domain in R^n with smooth boundary.

It is known that the following problems are still open:

1. Whether or not $u(x,t)$ blows up at finite time or not.

2. Identification of the set of blow–up points (if any).

3. The asymptotic behavior at the blow–up.

4. Existence (in a weak sense) of the solution beyond the blow–up point(s).

Remark. The Ω unbounded the case is, generally speaking, harder to deal with, see for example Levine [11] and Ni [13].

Remark. In this chapter, we deal with one–dimensional equation. In chapter three we deal with higher dimensional.

§2. Existence–Uniqueness of Steady State Solution

Consider the semilinear parabolic equation

$$\begin{aligned} &u_t = u_{xx} + f(x,u) \quad t, x > 0 \\ &u(x,0) = u_0 \\ &u(0,t) = h(t), \quad h(t) \text{ is continuous} \end{aligned} \tag{2.2.1}$$

Definition (2.2.2): v, w are respectively lower and upper solutions of (2.2.1) if they are as defined in Definition (1.1.3), (1.1.6) with

$$v(0,t) \le h(t) \le w(0,t) \quad t \ge 0$$

Consider the semilinear parabolic equation

$$\begin{aligned} &u_t = u_{xx} + f(u,x) - \tfrac{2u_x^2}{u+1} \quad x > 0, \;\; t > 0 \\ &u(x,0) = \varphi(x) \\ &u(0,t) = 0 \end{aligned} \tag{2.2.3}$$

where

(i) $f(u,x) > 0$ for $u_1, x > 0$

(ii) $\varphi(x) > 0$ for $x > 0$

If we set

$$v = \frac{1}{u+1} \tag{2.2.4}$$

Then (2.2.1) becomes

$$\begin{aligned} &v_t = v_{xx} - v^2 f(\tfrac{1}{v} - 1, x) \\ &v(x,0) = \tfrac{1}{\varphi(x)+1} \\ &v(0,t) = 1 \end{aligned} \tag{2.2.5}$$

Lemma (2.2.6) If $\lim_{v \to 0} v^2 f(\frac{1}{v} - 1, x) = 0$, then

$$0 < u(x,t) \leq \infty \qquad \text{for } x, t \geq 0 \tag{2.2.7}$$

Proof: By the above condition, $v \equiv 0$ is a lower solution of (2.2.5), i.e $u < \infty$. And $v \equiv 1$ is an upper solution for (2.2.5), i.e,

$$v^2 f(\frac{1}{v} - 1, x) = -v_t + v_{xx} \geq 0, \quad v/\partial\Omega \leq 1$$

$0 < v < 1$, which gives $u \geq 0$.

Lemma (2.2.8) If

(i) $F(u,x) \equiv \frac{1}{u+1} f(u,x)$ is continuous for $x > 0$; $u \geq 0$.

(ii) $F(u,x) > 0$ for $x \geq 0$, $u > 0$.

(iii) $F(u,x) < F(v,x)$ for $0 < u < v < \infty$, $x \geq 0$.

Then the steady state solution of (2.2.5) exists i.e. a solution of

$$\begin{aligned} &v_{\infty_{xx}} = v_\infty^2 f\left(\frac{1}{v_\infty} - 1, x\right) \\ &v_\infty(0) = 1 \end{aligned} \tag{2.2.9}$$

and it is monotonically decreasing.

A result by P. Wong [13] shows that v_∞ is monotonically decreasing.

Remark (2.2.10). The steady state solution of (2.2.5) i.e.

$$u_{\infty xx} + f(u_\infty, x) - \frac{2u^2_{\infty x}}{(1+u)} = 0 \qquad x \geq 0t > 0 \tag{2.2.11}$$

$$u_\infty(0) = 0$$

exists and monotonically increasing.

To prove this we need the following lemmas:

Lemma (2.2.12) Uniqueness. Let v be a nonnegative solution of (2.2.9) passing through $(0,1)$ and (b,B). Then the solution is unique.

Proof. Suppose that $w(x)$ is a second nonnegative solution such that $v(0) = w(0) = 1$ and $v(b) = w(b) = B$. We assume that $(0,1)$ and (b,B) are two consecutive points of intersection of v and w and $v(x) > w(x)$ for $0 < x < b$. Using (2.2.9) and property (iii) of (2.2.8), we find

$$\int_a^b \{v_{xx}w - vw_{xx}\}dx = \int_a^b wv[F(v,x) - F(w,x)dx \tag{2.2.13}$$

Since

$$\int_a^b \{v_{xx}w - vw_{xx}\}dx = B[v_x(b) - w_x(b)] - A[v_x(a) - w_x(a)] \tag{2.2.14}$$

and since $v_x(0) > w_x(0)$ while $v)x(b) < w_x(b)$ the right hand side of (2.2.14) is clearly negative which contradicts (2.2.13). If v and w showed other points of intersection on $(0, b)$, we can partition the interval $[0, b]$ into several segments whose endpoints are of the consecutive points of intersection of v and w. The same argument leads to contradiction in each case. This proves the assertion.

Lemma (2.2.15). Let $v(x)$ be a nonnegative solution of (2.2.9) passing through $(0,1)$ such that $\lim_{x \to b} v_x(x) = 0$ where b may be finite or infinite. Then $v(x)$ is unique.

The proof is identical to that of Lemma (2.2.12) since the right hand side of (2.2.14) will also be negative under the present assumption.

Lemma (2.2.16) Let $(a, A), (b, B)$ be two points such that $a < b$ and $A, B > 0$. $(b - a)$ is small enough so that

$$H(b-a)^2 < \rho < 1 \tag{2.2.17}$$

and

$$L(x) > \int_0^b g(x,t) L(t)\, F(L,t) dt \tag{2.2.18}$$

where

$$L(x) \equiv \frac{A(t-x) + B(x-a)}{(b-a)}$$

and

$$g(x,t) = \begin{cases} \dfrac{(b-x)(t-a)}{b-a)} & t \leq x \\ \dfrac{(b-t)(x-a)}{(b-a)} & x \leq t \end{cases} \tag{2.2.19}$$

Proof. To establish the existence we replace the boundary value problem by the equivalent integral equation

$$V(x) = L(x) - \int_a^b g(x,t)v(t)F(v,t)dt,$$

where $L(x)$ and $g(x,t)$ are given respectively to solve by successive approximations, we introduce a sequence $\{V_k(x)\}$ of twice differentiable convex functions passing through $(0,1)$ and (b,B) defined by

$$\begin{cases} V_0(x) = L(x) \\ V_{k+1}(x) = L(x) - \int_a^b g(x,t)V_k(t)F(v_k,t)dt. \\ k = 0,1,2,\ldots. \end{cases} \tag{2.2.20}$$

Since both $g(x,t)$ and $L(x)$ are positive in $(0,b)$, (2.2.14) shows that $0 < V_1(x) < L(x)$. if we assume that $0 < V_k(x) < L(x)$, then (2.2.14) and property (iii) of Lemma (2.2.8) implies

$$\begin{aligned} L(x) > V_{n+1}(x) &= L(x) - \int_a^b g(x,t)V_k(t)F(V_k,t)dt \\ &> L(x) - \int_a^b g(x,t)L(t)F(L,t)dt = V_1(x) > 0 \end{aligned}$$

It follows by induction that $0 < V_k(x) \leq L(x) \leq \max(1,B)$ for all k. The sequence $\{V_k(x)\}$ is thus positive and uniformly bounded. Let $K = \max(1,B)$ and $M = \sup F(k,x)$, then

$$\begin{aligned} |V_1(x) - V_0(x)| &= \int_a^b g(x,t)L(t)F(L,t)dt \\ &\leq KM \int_a^b g(x,t)dt \\ &\leq KM(b^2) \end{aligned}$$

If R denotes the closed rectangle defined by $0 \leq V \leq K$ and $0 \leq x \leq b$, then by

$$|ZF(z,x) - WF(w,x)| \leq H|z-w|$$

for all points of R. Moreover, (2.2.20) shows that

$$|V_{k+1}(x) - V_k(x)| \leq H \int_a^b g(x,t)|V_k(t) - V_{k-1}(t)|dt$$

so that we have, by induction,

$$|V_{k+1}(x) - V_l(x)| \leq (KM)H^*(b)^{2(k+1)}. \tag{2.2.21}$$

We thus obtain the estimate

$$|V_n(x)| \leq K + H^{-1}KM\sum_1^n [H(b)]^{k+1}$$

which, in view of (1.6), implies the uniform convergence of $\{V_n(x)\}$. This proves the lemma.

Lemma (2.2.22): There exists a unique positive solution $z(x)$ of equation (2.2.9) which passes through the two points $(0,1)$ and (b,B), where $b > 0$ and $1, B > 0$. If w denotes any other positive function of $D'[0,b]$ for which $w(0) = 1, w(b) = B$, and if $J(V_\infty; b)$ denotes the functional

$$J(V_\infty; b) = \int_a^b [(V_{\infty x})^2 + 2h(V_\infty, x)dx, \tag{2.2.23}$$

where

$$h(V,x) = \int_a^v +g(t,x)dt \tag{2.2.24}$$

Then

$$J(z;b) < J(w;b) \tag{2.2.25}$$

unless $w(x) \equiv z(x)$ in $[0, b]$.

We first assume that the interval $[0, b]$ is short enough so that conditions (2.2.17) and (2.2.14) are satisfied. Lemma (2.2.16) will then guarantee the existence of the unique positive solution z of (2.2.9) through the two points, and all we have to prove is inequality (2.2.21). To do so, we note that the solution $h(x)$ of the linear differential system

$$\begin{cases} h'' = P(x)h \quad P(x) > 0 \\ h(a) = A \\ h(b) = B \end{cases} \tag{2.2.26}$$

satisfies the inequality

$$\int_a^b [(h')^2 + P(x)h^2]dx < \int_a^b [(w')^2 + P(x)w^2]dx, \tag{2.2.27}$$

where w is any other function of $D'[0, b]$ which satisfies the same boundary conditions and does not coincide with $h(x)$. Inequality (2.2.27) is an obvious consequence of the identity

$$\begin{aligned} &\int_a^b [(w' - h')^2 + P(x)(w - h)^2]dx \\ &\quad = \int_a^b [(w')^2 + P(x)w^2]dx - \int_a^b [(h')^2 + P(x)h^2]dx \end{aligned}$$

which is obtained by expanding the left–hand side and observing that in view of

(2.2.25 and the boundary conditions,

$$\int_a^b w'h'dx = [wh']_a^b - \int_a^b wh'dx = [wh']_a^b - \int_a^b Pwhdx.$$

and

$$[wh']_a^b = [hh']_a^b = \int_a^b (h'^2 + hh'')dx = \int_a^b (h'^2 + Ph^2)dx.$$

Setting, in particular, $P(x) = F(z,x)$, we have $h(x) = z(x)$ and thus by (2.2.27)

$$\int_a^b [(z')^2 + z^2F(z,x)]dx < \int_a^b [(w')^2 + w^2F(z,x)]dx. \tag{2.2.28}$$

Since $F(s,x)$ is a nondecreasing function of s for $s > 0$, the function $E(z,x)$ defined by (2.2.24) is convex in z. Hence, for nonnegative z and w,

$$2[E(z,x) - E(w,x)] \leq (z^2 - w^2)F(z,x).$$

Combining this with (2.2.28), we obtain

$$\int_a^b [(z')^2 + 2E(z,x)]dx < \int_a^b [(w')^2 + 2E(w,x)]dx.$$

unless z and w coincide. This establishes (2.2.25) in the case in which the interval $[0,b]$ is short enough so as to satisfy conditions (2.2.17) and (2.2.14).

If b is an arbitrary value in $(0,\infty)$, it is sufficient to consider the problem

$$\begin{cases} J(V_\infty; b) = \int_a^b [(V_{\infty x})^2 + 2E(V_\infty, x)]dx = \min \\ V_\infty(0) = A, \quad V_\infty(b) = B. \end{cases} \tag{2.2.29}$$

in the class Ω_b of nonnegative convex functions $V_\infty \in D'[0,b]$. We thus may assume

$$0 \leq V_\infty(x) \leq \max(1,B) = k, \quad 0 \leq x \leq b.$$

Now we divide the interval $[0, b]$ into a finite number of subintervals $[a_k, a_{k+1}]$ $(0 = a_0 < a_1 < \cdots < a_m = b)$ in each of which the assumptions of Lemma (2.2.16) are satisfied. If $V_\infty(a_k) = A_k$ where $V_\infty \in \Omega_b$, the conditions restricting the length of these subintervals will be

$$H(a_{k+1} - a_k)^2 < P < 1 \tag{2.2.30}$$

and

$$\int_{a_k}^{a_{k+1}} g_k(x,t) L_k(t) F(L_k, t) dt < L_k(x). \tag{2.3.31}$$

where

$$L_k(x) = \frac{A_k(a_{k-1} - x) + A_{k+1}(x - a_k)}{(a_{k-1} - a_k)} \tag{2.2.32}$$

and

$$g_k(x,t) = \begin{cases} \dfrac{(a_{k+1} - x)(t - a_k)}{(a_{k+1} - a_k)}, t \leq x \\ \dfrac{(a_{k-1} - t)(x - a_k)}{(a_{k+1} - a_k)}, x \leq t. \end{cases} \tag{2.2.33}$$

Since $A_k \leq \max(1, B) = K$, we have $F[L_k(t), t] < F(K, t)$. Hence, if $M = \max F(K, x)$ in $[a, b]$, condition (2.2.31) will be satisfied if

$$M \int_{a_k}^{a_{k+1}} g_k(x,t) L_k(t) dt < L_k(x).$$

In view of (2.2.32), this will be true if both the inequalities

$$M \int_{a_k}^{a_{k+1}} g_k(x,t)(a_{k-1} - t) dt < (a_{k+1} - x)$$

and

$$M \int_{a_k}^{a_{k+1}} g_k(x,t)(t - a_k) dt < (x - a_k) \tag{2.2.34}$$

hold. Since these inequalities are equivalent, it is sufficient to consider one of them. A computation shows that

$$\int_{a_k}^{a_{k+1}} g_k(x,t)(t-a_k)dt = \frac{1}{6}(x-a_k)(a_{k+1}-x)(x+a_{k+1}-2a_k),$$

and (2.2.34) will therefore follow if

$$\frac{M}{6}(a_{k+1}-x)(x+a_{k+1}-2a_k) < 1.$$

Since

$$(a_{k+1}-x) \leq (a_{k+1}-a_k)$$

and $(x+a_{k+1}-2a_k) = (x-a_k)+(a_{k+1}-a_k) \leq 2(a_{k+1}-a_k)$ the length of the interval is thus restricted by the condition

$$M(a_{k+1}-a_k) < 2$$

and inequality (2.2.30). Since $H = H(K,a,b)$, this shows that a finite partition of the type indicated is indeed possible.

In each of these subintervals we now replace $V_\infty, V_\infty \in \Omega_b$, by the solution (2.2.9) having the same values at the ends of the interval. If the new function so obtained is V_∞^*, it follows from the result just proved that

$$J(V_\infty^*; b) < J(V_\infty; b).$$

In the treatment of the minimum problem (2.2.29) it is therefore sufficient to consider curves V_∞ consisting of a finite number of arcs each of which is a solution of (2.2.9). Moreover, the abscissas of the points where two adjacent arcs meet may be taken to be the same for all functions of a sequence $\{y_n\}$ minimizing the functional $J(V_\infty; b)$.

Since in each of the subintervals $[a_k, a_{k+1}]$ the functions y_n are solutions of (2.2.9), elementary considerations show that we can select a subsequence $\{y_n\}$ which converges in each subinterval $[a_k, a_{k+1}]$ to a solution $y_{(k)}$ of (2.2.9) and that, moreover, $y_{(k)}(a_{k+1}) - y_{(k+1)}(a_{k+1})$. The function V_∞ defined by $y(x) = y_{(k)}(x)$ for $a_k \leq x \leq a_{k+1}$ is therefore of class $D'[0, b]$, and it is thus a solution of the minimum problem (2.2.29).

To show that $V_\infty(x)$ coincides in all these intervals with the same solution of (2.2.9), we have to show that $V_{\infty x}$ is continuous at the points a_k. To do so, we choose a positive ϵ such that

$$a_{k-1} + \epsilon) < a_k, \quad (a + \epsilon) < a_{k+1}$$

and ϵ is small enough so that Lemma (2.2.16) applies to the interval $[a_k - \epsilon, a_k + \epsilon]$. There will then exist a solution z of (2.2.9) for which $z(a_k - \epsilon) = V_\infty(a_k - \epsilon), (a_k + \epsilon) = y(ak + \epsilon)$ and, as shown above, we have the inequality

$$\int_{a_{k-\epsilon}}^{a_{k+\epsilon}} +2[(zx)dx < \int_{a_{k-\epsilon}}^{a_{k+\epsilon}} [V_{\infty x}^2 + 2[(V_\infty, x)dx$$

unless $V_\infty(x) \equiv Z(x)$ in this interval. Hence if $V_{\infty x}$ is discontinuous at $x = a_k$, it is possible to replace V_∞ by another function which yields a smaller value of $J(V_\infty; b)$. But this contradicts the minimum property of V_∞, and we have thus proved that $V'_{\infty x}$ must be continuous throughout $[0, b]$. This completes the proof of Lemma (2.2.22).

Theorem (2.2.35) For any given (a, A) when $A > 0$ there exists exactly one positive solution u_∞ of (2.1.8) of the class $G^2[a, \infty)$ which passes through (a, A) and is monotonically increasing in (a, ∞).

Proof. To prove this result, we prove the equivalent result for (2.2.11), consider

$$J(v) = \int_a^\infty [(v_x)^2 + 2h(v, x)dx \qquad (2.2.36)$$

within the class π of all nonnegative functions $v \in D'[a, \infty)$ such that $v(a) = A$ and the integral (2.2.35) exists since (2.2.9) is the Euler-Lagrange equation of the problem (2.2.36), the solution v of (2.2.36) will be a solution of (2.2.9) provided v exists and of class $C^2[a, \infty)$.

Since the functional $J(v)$ is positive–definite, $J(v)$ has the trivial lower bound 0.

We next remark that we restrict our attention to positive function $v \in \pi$ which are convex in $[a, \infty)$. To show this, we assume that the positive function v is convex in an interval (c, d), 2.e

$$v(x) \geq \frac{v(c)(d-x) + v(d)(x-c)}{(d-c)}$$

In view of hypothesis (iii) of Lemma (2.2.9), and definition of $h(v, x)$ we have

$$h(L, x) \leq h(v, x) \quad 0 \leq x \leq d$$

and, by a variational argument

$$\int_c^d [L'(x)]^2\,dx < \int_c^d (v_x)^2 dx \tag{2.2.37}$$

unless $v(x) \equiv L(x)$ in (c, d). Hence if v^* denotes the function obtained from v by substituting $L(x)$ for $v(x)$ in $c, d)$,

$$J(v^*) < J(v)$$

Also, we need only consider positive convex function c which are nonincreasing in $[a, \infty)$, since as (2.2.36) shows, the functional $J(v)$ becomes infinite for convex increasing function. Finally, the problem $J(v) = \min$ is not vacuous, since the function v defined by

$$v(x) = \begin{cases} A\left(\frac{b-x}{b-a}\right) a \leq x \leq b \\ 0 \qquad b \leq x \end{cases}$$

is in π and evidently

$$J(v) < C < +\infty$$

As lemma ensures the result for finite case.

It is sufficient to consider positive admissible functions $v \in \pi$ which are convex and decreasing in $[a, \infty)$ if v is any such function, we choose b in $[a, \infty)$ and define a function $w \in T, T^2 \subset \pi$ as follows: $w(x) = v(x)$ in $[a, \infty)$ and $w(x) = v_b(x)$ where $v_b(x)$ denotes the solution of (2.2.11) whose existence is established in Lemma (2.2.13), which satisfies $v_b(a) = A$ and $v_b(b) = v(b)$. In view of Lemma (2.2.13), we have

$$J(v_b) < J(v)$$

and it is clear that $0 \leq v_b(x) \leq A$ in (a, ∞). We now take a sequence $\{v_h\}$ in π for which

$$\lim J(v_n) = \inf J(v) \tag{2.2.38}$$

and we choose a sequence of value b_m $(b > b_1 < b_2 < \cdots)$ for which $\lim_m b_m = +\infty$. For each of these values b_n we construct the corresponding function $v_n, b_n \in T$. As just shown, we have

$$J(v_n, b_{m+1}) \leq J(v_n, b_m)$$

hence, the diagonal sequence $J(v_n, b_n)$ cannot have larger limit than the sequence $J(v_n)$ and (2.2.38) shows that $J(v_n, b_n)$ is likewise a minimizing sequence. Since $0 \leq v_n, b_n \leq A$ and since, v_n, b_n is a solution of (2.2.11) in $[a, b_n]$ if $n \geq N$ an elementary argument shows that this, say, contains a limit function v which is a solution of (2.2.11) in $[a, b_n]$. But N is arbitrary, and v is thus a solution of (2.2.11) throughout $[a, \infty)$ the function is necessary convex must be decreasing for $a \leq x < \infty$. This completes the proof of Theorem (2.2.35).

Theorem (2.2.39) Equation (2.2.9) has a solution which ultimately decreases monotonically to positive constants if and only if, there exist some $\beta > 0$ so that

$$\int^{\infty} xF(\beta, x)dx < +\infty \tag{2.2.40}$$

Proof. If v_∞ is such a solution, it is easily confirmed that

$$v_\infty(x) = v_\infty(b) + v_{\infty_x}(b)(x - b) + \int_x^t (t - x)v_\infty(t)F(v_\infty, t)dt$$

Since $v_z\infty(b) > 0$ and $v_{\infty_x}(b) < 0$, it follows that

$$v_\infty(a) \geq v_\infty(x) \geq \int_x^b (t-x)v_\infty(t)F(v_\infty,t)sdt$$
$$\geq \alpha \int_x^b (t-x)F(\alpha,t)dt$$

where $\lim v_\infty(x) = \alpha > 0$. This shows that condition (2.2.40) is necessary. To show sufficiency consider,

$$v_\infty(x) = \alpha + \int_x^\infty (t-x)v_\infty(t)F(v_z\infty,t)dt \tag{2.2.41}$$

and suppose $\beta > 0$ so that (2.2.40) holds. Then we can find a point $a \geq 0$ such that for all $x \geq a$, we have

$$\int^\infty (t-x)F(\beta,t)dt < \frac{1}{2}$$

We define a sequence $\{v_{\infty_x}(x)\}$ by

$$\begin{cases} v_{\infty_0}(x) = \alpha \\ y_{\infty_{k+1}}(x) = \alpha + \int_x^\infty (t-x)v_{\infty_k}(x)F(v_{\infty_k},t)dt \\ k = 0,1,2,\ldots \end{cases} \tag{2.2.42}$$

If we choose α such that $0 < \alpha < \beta/2$ we see that $0 < \alpha < v_{\infty_x}(x)$ and $v_{\infty_0}(x) = \alpha < \beta$. By assuming $v_{\infty_k}(x) < \beta$ we find that

$$v_{\infty_{k+1}}(x) = \alpha + \int_x^\infty (t-x)v_{\infty_k}(x)F(v_{\infty_k},t)dt$$
$$< \frac{\beta}{2} + \beta \int_x^\infty (t-x)F(\beta,t)dt < \beta.$$

Hence, induction shows that $0 < \alpha \leq v_{\infty_k}(x) < \beta$ for all k. Moreover, if x_1 and x_2 are any two points such that $a \leq x_1 < x_2 < \infty$, then from (2.2.42) we have

$$|v_{\infty_{k+1}}(x_2) - v_{\infty_{k+1}}(x_1)| \leq |x_2 - x_1| \times$$

$$\{\int_{x_1}^{x_2} v_{\infty_k} F(v_{\infty_k}, t)dt + \int_{x_1}^{\infty} v_{\infty_k} F(v_{\infty_k}, t)dt\}$$

$$= |x_2 - x_1| \int_{x_1}^{\infty} v_{\infty_k}(t) F(v_{\infty_k}, t)dt$$

In view of uniform boundedness of $\{v_{\infty_k}\}$ and (2.2.40), it follows that the sequence is equi–continuous also. Since $F(u, x) < F(v, x)$ whenever $0 < u < v < \infty$, it follows form the assumption $v_{\infty k+1} < v_{\infty k}$ that

$$v_{\infty_{k+2}}(x) - v_{\infty_{k+1}}(x) = \int_x^{\infty} (t - x) \left[v_{\infty_{k+1}} F(v_{\infty_{k+1}}, t) - v_{\infty_k} F(v_{\infty_k}, t) \right] \quad dt > 0$$

This, together with the fact that $v_{\infty_1} > v_{\infty_{(1)}}$ shows that $\{v_{\infty_k}(x)\}$ is a monotonically increasing sequence. We can therefore find a uniformly converging subsequence whose limit function $v(x)$ is the solution of equation (2.2.41).

It remains to show that the solution of (2.2.42) so obtained is indeed of class $C^2[0, \infty)$ and satisfies (2.2.9). To this end, we observe that, for $h > 0$,

$$\left| \frac{v_\infty(x + h) - v_\infty(x)}{h} + \int_x^{\infty} v_\infty(t) F(v_\infty, t)dt \right|$$

$$\le \int_x^{x+h} \left| \frac{x - t}{h} \right| v_\infty(t) + F(v_\infty, t)dt$$

$$\le \int_x^{x+h} v_\infty(t) F(v_\infty, t)dt$$

A corresponding inequality holds for $h < 0$. The solution v_∞ of (2.2.41) being continuous in $[0, \infty)$, it follows that

$$v_{\infty_x}(x) = -\int_x^{\infty} v_\infty(t) F(v_\infty, t)dt.$$

In a similar manner, $v_{\infty_x} = v_\infty F(v_\infty, x)$ can be shown and the conclusion follows.

Theorem (2.2.43) Equation (2.2.9) has a unique solution which blows–up at $(x,t) = (\infty,\infty)$ if and only if, for each $\beta > 0$

$$\int^{\infty} xF(\beta, x)dx = +\infty \tag{2.2.44}$$

with $v(x,t) \to v_\infty(x)$ are less than exponential.

<u>Proof</u> We note that Theorem (2.2.35) assures the existence of a positive solution of (2.2.9) which is asymptotically equivalent to either a positive constant or zero and Theorem (2.2.39) gives a condition which is necessary and sufficient. it follows that (2.2.44) is both necessary and sufficient for a solution to decrease to zero (i.e.) for a steady state solution u_∞ to blow–up at ∞.

Now, let v_∞ be the steady state, as above, we would like to see that the asymptotic behavior of $v(t,x) \to v_\infty(x)$, as $t \to \infty$. Let $w = v_\infty + C(t)r(x)$ we would choose $C(t), r(x) > 0$ so as to have w as upper solution

$$w_t \le w_{xx} - \tilde{F}(w,x), \quad \tilde{F} \equiv w^2 f(\frac{1}{w} - 1, x)$$

$$c'(t)r(x) \le v_{\infty_{xx}} + c(t)r''(x) - \tilde{F})w, x)$$

$$= \tilde{F}(v_\infty, x) - \tilde{F}(w,x) + c(t)r''(x)$$

Let $\tilde{F}$ satisfy the intermediate–value theorem

$$c'(t) \le \{-\tilde{F}_{v\infty}(\eta(x), x) + \frac{r''(x)}{r(\alpha)}\}c(t)r(x)$$

By (iii) of Lemma (2.2.8)

$$-\tilde{F}_{v_\infty}(\eta(x), x) < 0.$$

$$s_t = \{\eta : v_\infty \leq \eta \leq v_\infty + c(t)r\}$$

So $r(x)$ can be chosen so that

$$a_i \equiv \sup_{\substack{\eta \in S_t \\ x}} \left\{ -\tilde{F}_{v\infty}(\eta(\alpha), x) + \frac{r''(x)}{r(x)} \right\} < 0.$$

So

$$c(t) = c(0)\, e^{\int_0^t a_s d_s} \leq c(0) e^{a_0 t} \quad \text{for all } t,$$

$$a(t) > c(t) > 0 \Rightarrow a_t < a_0 < 0$$

§3. Examples

Example (2.3.1)

If $f(u,x) = (u+1)^\lambda \quad 1 < \lambda < 2$, we have $\lim\limits_{v \to 0} v^2 \cdot \left(\frac{1}{v}\right)^\lambda = 0$ and $\int^\infty x\beta \left(\frac{1}{\beta}\right)^\lambda dx =$ $+\infty$ for each $\beta > 0$. Thus by Theorem (2.2.9) we have

$$u(x,t) \to \infty \quad \text{as} \quad x, t \to \infty \quad \text{of} \quad u_t = u_{xx} + (u+1)^\lambda - \frac{2u_x^2}{u+1}$$

or $u(x,t)$ blows up at $(x,+) = (\infty, \infty)$.

Example (2.3.2)

If $f(u,x) = (u+1)\log(u+2)$, $\lim\limits_{v \to 0} v^2 \cdot \left\{ \frac{1}{v} \log\left(\frac{1+v}{v}\right) \right\} = 0$ and, $\int^\infty x \ln\left(\frac{\beta+1}{\beta}\right) dx =$ ∞ for each $\beta > 0$. Thus by Theorem (2.2.9), $u(x,t)$ blows up at $(x,t) = (\infty, \infty)$ of

$$u_t = u_{xx} + (1+u)\log(u+2) - \frac{2u_x^2}{(u+1)} \quad x, t > 0$$

$$u(0,x) = u_0 \quad \text{suitable one}$$

$$u(t,0) = 0$$

§4. Regularly Varying Functions

In this section we study the asymptotic behavior of $v_\infty(x)$ as $x \to \infty$. For that we need

Definition (2.4.1)

A function $\phi(x)$ is regularly varying function with exponent σ iff ϕ is measurable and eventually positive and

$$\phi(xy)/\phi(x) \to y^\sigma \quad \text{as } x \to \infty$$

for $y > 0$.

Notation: Regularly varying functions with exponent σ are denoted by RV_σ.

Lemma (2.4.2)

Let $F(v,x) = v^\lambda \phi(x), \quad \lambda > 1$. Suppose $f \in RV_\sigma$. Then there exists two functions $f_1 \sim f_2$ such that

$$f_1(t) \le f(t) \le f_2(t) \quad \text{for } t \ge t_0$$

and such that the functions $\psi_i \equiv \log f_i(e^i)$ are C^∞ on a neighborhood of ∞ and satisfies

$$\psi_i'(\tau) \to \alpha \quad \text{as } \tau \to \infty$$

and

$$\psi(n)_i(\tau) \to \alpha \quad \text{as } \tau \to \infty$$

for $i = 1, 2$.

Lemma (2.4.3)

Let $F_1(v, x) = \phi(x)v^\lambda$, $\lambda > 0$ and y be bounded positive solution of

$$v'' = \phi(x)v^\lambda, \quad \lambda > 1, \;\; x > 0$$

$$v(0) = 1 \tag{2.4.4}$$

of $\phi(x) \in RV_\sigma$ with $\sigma > -2$, then

$$v \in RV_{-(\sigma+2)/(\lambda-1)}.$$

Proof: Substitution of $u = y^{1-\lambda}$ and $V(x) = \log u(e^x)$ shows that V satisfies the equation

$$V'' - V' - \beta V'^2 = -e^{\psi - v}, \tag{2.4.5}$$

where $\psi = \psi(x)\log\{(\lambda - 1)e^{2x}\phi(e^x)\}$ and $\beta = (\lambda - 1)^{-1} > 0$.

By Lemma (2.4.3), there exists ψ, with $\psi - \psi_1 \to 0, \psi_1' \to 6 + 2$, $\psi_1'' \to 0$, and $\psi_1 \leq \psi$ for x sufficiently large. Substituting $V = \psi_1 + c$ in (2.4.5) now gives

$$c'' + yc' - \beta c'^2 = (-1 + 0(1))e^{-c} + (6 + 2)(1 + \beta 6 + 2\beta) + 0(1), \tag{2.4.6}$$

with $y :\equiv y(x) \to 2\beta(6+2)+1 (x \to \infty)$. We claim that $c = c(x)$ tends to a finite limit as $x \to \infty$.

The following three cases are possible:

(i) $c' > 0$ for $x > x_0$. Then c is ultimately increasing, i.e., $\lim c(x) \leq \infty$ exists. If $c(x) \to \infty$, then by (2.4.6) for x sufficiently large, $\sigma' > x\delta + \beta\delta^2 > \frac{1}{2}\sigma$, where $\delta := c'$. This implies $\delta \to \infty$ and by (2.4.6), $-1/\sigma)' = \delta'/\delta^2 \to \beta$; hence, $-1/c' = -1/\delta \sim \beta x (x \to \infty)$. This contradicts the assumption $c' > 0$.

(ii) There is a sequence $x_k \to \infty (k \to \infty)$ with $c'(x_k) = 0$. Assume x_k is the sequence of all consecutive zeros of c. If $c''(xx_k) < 0$ (c attains its maximum in x_k) and $\epsilon > 0$ arbitrary, then $c(x_k) < -\log\{(\sigma + 2)(1 + \beta\sigma + 2\beta)\}$ for large k, by (2.4.6). Similarly, if $c''(x_k) > 0$ we find $c(x_k) > -\log\{(\sigma + 2)(1 + 2\beta\sigma + 2\beta)\}$; hence, a contradiction.

(iii) $c' < 0$ for $x > x_0$. Then c is ultimately decreasing. If $c(x) \to \infty$ $(x \to \infty)$, then, since $\psi_1 \leq \psi$, we have using (2.4.5)

$$-V'' + V' + \beta V'^2 = e^{\psi - v} \geq e^{\psi_1 - v} = e^{-c}$$

Hence there exists a sequence $x_n \to \infty$ $(n \to \infty)$ such that $V'(x_n) \to \pm\infty$. If $V'(x_n) \to +\infty$, then $c'(x_n) \to +\infty$; hence, a contradiction. The case $V'(x_n) \to -\infty$ implies $u'(\exp x_n) < 0$ hence, $y'(\exp x_n) > 0$ for n sufficiently large Since $y'' > 0$, this contradicts the boundedness of y. This finishes the proof, since $\psi - v \to$ constant implies $x^2\phi(x) \sim cy^{1-\lambda}(x)$; hence y is regularly varying.

Theorem (2.4.7)

Let the right hand side of equation (2.1.7) be $v_\infty^2\phi(x)$. If $\phi(x) \in RV_\sigma$, $\sigma > -2$, $\lambda > 1$, $v_\infty(x) \to 0$ at ∞, like

$$v_\infty(x) \sim \left[\frac{(1+\lambda+\sigma)(\sigma+2)}{(\lambda-1)^2}\right]^{\frac{1}{\lambda-1}} \{x^2\phi(x)\}^{-1(\lambda-1)} \quad \text{as } x \to \infty. \tag{2.4.8}$$

Proof: By Lemma (2.4.3), $x^2 v_\infty(x) \sim c_0 v_\infty(x)$ with $c_0 = (\sigma+2)(\sigma+1+\lambda)/(1-\lambda)^2$. Substituting this in equation (2.2.9) gives (2.4.8).

Corollary (2.4.9).

If $f(u,x) = \phi(x)(u+1)^{2-\lambda}$, $\phi(x) \in RV_\sigma$ $\sigma < -2$, $\lambda > 1$, then

$$u_\infty \ \sigma\{x^2\phi(x)\}^{\frac{1}{(\lambda-1)}} \left[\frac{(1+\lambda+\sigma)(\sigma+2)}{(\lambda-1)^2}\right]^{-\frac{1}{(\lambda-1)}}$$

Remark: This material was based on a paper by Geluk [9].

§5. Examples, using Regularly Varying Functions

Example (2.5.1)

Let $f(u,x) = \phi(x)(u+1)^{2-\lambda}$, $\lambda > 1$. Then easy verification shows that Theorem

(1.8.1) holds for corresponding function i.e.

$$v^2 f(\frac{1}{v} - 1, x) = \phi(x) v^{\lambda-2}$$

$$F(v, x) v f(\frac{1}{v} - 1, x) = \phi(x) v^{\lambda-1}$$

if $\phi(x) \in RV_\sigma$ with $\sigma > -2$, then

$$\int^{\infty} x\phi(x)\beta^{\lambda-1} dx = +\infty \qquad \text{for each } \beta > 0$$

Thus $v_\infty \to 0$. Applying Theorem (2.4.6) gives that $u_\infty(x)$ tends to ∞ like

$$\left[\frac{(1+\sigma+\lambda)(\sigma+2)}{(\lambda-1)^2}\right]^{-\frac{1}{(\lambda-1)}} \cdot (x^2\phi(x))^{\frac{1}{(\lambda-1)}}.$$

Example (2.5.2).

In the above example if we put $\sigma = -3/2$ $\lambda = 3/2$, then $u_\infty(x) \sim \{x^2\phi(x)\}^2$ as $x \to \infty$. Note that $\phi(x) \in RV_\sigma$ $\sigma < -2$.

Chapter III Higher Dimensions Reaction–Diffusion Equations

§1. Recent Developments.

Consider the semilinear parabolic equation

$$u_t = \Delta u + u^p \quad x \in R^n, \quad t > 0, \ p > 1, \ u \geq 0$$

$$u_{t=0} = \phi \in C_B(R^n) \quad \phi \geq 0, \ \phi \not\equiv 0. \tag{3.1.1}$$

if $p > \dfrac{n+2}{n}$ and for any $k > 0$ there exists small $\delta = d(k, p, n)$ such that

$$0 \leq \phi(x) \leq \delta(k + |x|^2)^{-1/(p-1)}$$

then (3.1.1) has global classical solution decaying like $t^{-1/p-1)}$ as $t \to \infty$.

Furthermore, if

$$\lim_{|x| \to \infty} \inf |x|^{2/p-1} \phi(x) \leq (\lambda(B_1))^{1/(p-1)}$$

where B_1 is the unit ball on R^n and $\lambda(B_1)$ is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition. Then the smallest decay of $\phi(x)$ for global existence is $|x|^{-2/p-1}$ at ∞.

This result was given recently by Lee and Ni [10]. Another recent result obtained by Wong [14] is the following:

If $n \geq 3$ and $p > n/(n-2)$

$$0 \leq \phi(x) \leq u_s(x) = \lambda \left(\frac{2(n-2)}{(p-1)^2}\left(p - \frac{n}{n-2}\right)\right)^{1/(p-1} |x|^{-2/(p-1)}$$

(u_s is a singular equilibrium of (3.1.1)) where $0 < \lambda < 1$ then (3.1.1) has unique global solution u with

$$0 \leq u \leq \lambda u_s$$

and

$$u(x,t) \leq (\lambda^{1-p} - 1)(p-1)t)^{-1/(p-1)}.$$

Remark. In the coming sections, we will study similar problems with some term added.

§2. Lower and Upper Solutions

Consider the semilinear equation

$$\begin{aligned} u_t &= \Delta u + f(u,x) - \frac{2|\nabla u|^2}{u+1} \quad x \in R^n, \quad t > 0 \\ u(x,0) &= \varphi(x), \varphi(x) \text{ is continuous, } x \in R^n \end{aligned} \tag{3.2.1}$$

under the change of variable $v = \frac{1}{u+1}$ becomes

$$\begin{aligned} v_t &= \Delta v - F(v,x) \quad x \in R^n, \quad t > 0 \\ v(x,0) &= v_0 = \frac{1}{1+\varphi(x)} \quad x \in R^n \end{aligned} \tag{3.2.2}$$

One can easily show that $\underline{v} = 0$ and $\overline{v} = 1$ are lower and upper solution of (3.1.2) if $\lim_{v \to 0} F(v,x) = 0$ and all the conditions satisfying Theorem (1.1.8).

§3. Existence of Steady–State Solution

In this section the results used in existence of steady–state solution of (3.2.2) i.e. solution of

$$\Delta v = F(v, x) \quad x \in R^n \tag{3.3.1}$$

ar given by Ni [10].

What follows $x = (x_1, x_2) \in R^{n-m} \times R^m$.

Theorem (3.3.2)

Let $F(0, x) = v^p K(x) \quad p > 1$ or $p = \frac{n+2}{n-2}$, $K(x)$ is bounded locally Hölder continuous in R^n. If $m \geq 3$ and

$$|K(x_1, x_2)| < \frac{C}{|x_2|^\ell}$$

for large $|x_2|$, uniformly in $x_1 \in R^{n-m}$ for some constant $C > 0$, $\ell \leq 2$, then (3.3.1) possesses infinite number of bounded positive solution and tends to zero at infinite in x_2–direction.

Theorem 3.3.3.

Consider the equation

$$\Delta v = F(v, x) \tag{3.3.4}$$

in R^n where $x = (x_1, x_2) \in R^{n-m} \times R^n$ and F in locally Hölder continuous with the following properties

(i) there exist constant $C > 0, \ell \leq 2$ and $\rho > 0$ such that

$$F(x,p) \leq \frac{C}{|x_2|^\ell}$$

for x_2 large uniformly in $x_1 \in R^{n-m}$ where $m \geq 3$ (3.3.5)

(ii) there exist constant $c > 0, \delta > 0$ and $p > 1$ such that $F(x,\beta) \leq (c/\rho)^p F(x,p)$ uniformly in $x \in R^n$ and for all $\beta \in (0,\delta)$. (3.3.6)

(iii) $F(x,\beta) \geq 0$ for all $\beta \in (0,\delta)$ and $x \in R^n$. (3.3.7)

Then (3.3.4) possesses infinite number of positive bounded solutions such that $v \to 0$ in x_2-direction.

§4. Decaying Solution of v equation

Let $v_\infty(x)$ be a solution of (3.3.1). Consider

$$W \equiv v_\infty(x) + c(t)\varphi(x) \quad c(t) > 0, \;\; \varphi(x) > 0$$

to be an upper solution of (3.2.1), i.e.,

$$W_t \leq \Delta W - F(W,x) \quad x \in R^n, \quad t > 0$$

$$c'(t)\varphi(x) \leq \Delta v_\infty(x) + c(t)\Delta\varphi(x) - F(W,x)$$

$$= F(v_\infty, x) - F(W,x) + c(t)\varphi(x)$$

Let the intermediate theorem hold for $F(W,x)$ w.r.t. W. We have

$$c'(t) \leq \left\{ -\frac{\partial F}{\partial v_\infty}(\eta(x),x) + \frac{\Delta\varphi}{\varphi} \right\} c(t)$$

Since $F(v,x)$ is increasing on v, thus $-\frac{\partial F}{\partial v_\infty}(\eta(x),x)$ is negative, so $\frac{\Delta\varphi}{\varphi}$ can be chosen such that

$$S_t = \{\eta :\leq v_\infty + c(t)\varphi\}$$

$$a_t \equiv \sup_{\substack{\eta\in S_t \\ x}} \left\{ -\frac{\partial F}{\partial v_\infty}(\eta(x),x) + \frac{\Delta\varphi}{\varphi} \right\} < 0. \tag{3.4.1}$$

$$c(t) = c(0)e^{\int_0^t a_s ds}$$

$$\leq c(0)e^{a_0 t} \quad \text{for all } t > 0$$

$$c(0) > c(t) > 0 \Rightarrow a(t) < a(0) < 0$$

Theorem (3.4.2)

Let conditions of Theorem (1.1.8) hold, choose $\varphi(x)$ such that (3.4.1) holds. Then $v(x,t) \to v_\infty(x)$ as $e^{a_0 t}$. Furthermore, if $F(v,x)$ satisfies conditions of Theorem (3.3.3), then $v(x,t) \to 0$ in x_2–direction which means $u(x,t) \to \infty$ in x_2–direction as $x,t \to \infty$.

Proof: $v(x,t) \to v_\infty(x)$ is obvious by the above argument. $v(x,t) \to 0$ in x_2–direction as $x,t \to \infty$ is obvious by Theorem (3.3.3).

Corollary (3.4.3)

Under the conditions of Theorem (3.4.2) solution $u(x,t)$ of (3.2.1) blows up at ∞, in x_2–direction i.e., $u(x,t) \to \infty$ as $x,t \to \infty$ in x_2–direction.

§5. Examples

Example (3.5.1) Let

$$F(v,x) = \frac{v^{\alpha}}{(1+|x|^2)} \quad n \geq 4, \quad 0 < \alpha < 1$$

Then F satisfies all condition of Theorem (1.1.8). Theorem (3.3.3) and (3.4.1) also hold. Thus if $x = (x_1, x_2)$ where $x_2 \in R^m$, $m \leq 3$. Then $v(t,x) \to v_{\infty}(x) \to 0$ $t \to \infty$, $|x| \to \infty$ in x_2–direction which means the solution u of

$$u_t = \Delta u + \frac{(u+1)^{2-\alpha}}{(1+|x|^2)} - \frac{2|\nabla u|^2}{u+1} \qquad x \in R^4, \quad t > 0$$

$$u|_{t=0} = \phi \text{ with suitable } \phi$$

$u(x,t) \to \infty$ in x_2–direction as $t, x \to \infty$.

Example (3.5.2).

Let

$$F(v,x) = \frac{\ln(v+2)}{(1+|x|^2)} \quad n \geq 4, \quad x = (x_1, x_2) \in R^n \tag{3.5.3}$$

This satisfies condition of Theorem (1.1.8), and (4.1.1) and hence, the solution of

$$u_t = \Delta u + \ln\left(\frac{2u+3}{u+1}\right)(u+1)^2 - \frac{2|\nabla u|^2}{(u+1)}$$

$$u|_{t=0} = \phi \qquad \text{with suitable } \phi$$

blows up at ∞ in direction of x_2.

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